Name Abdulla Khalil, ID 89776

Exam Two, MTH 213 , Summer 2022

Ayman Badawi

Score = $\frac{43}{44}$ Excellent

Well organized
Nice handwriting

QUESTION 1. (6 points) Consider the following code C

For $k = 3$ to $(n^2 + 1)$
$w = 3 * k^4 + 7 * k^2 - 12$ = 8
For $i = 2$ to $(2k + 4)$
$y = 3 * k + 5 * i^3 + 2 * k - 21$ = 8
next i
next k

(i) Find the exact number of the arithmetic operations that are executed by the given code.

Outer Loop:

- Iterates $(n^2+1)-3+1 = n^2-1$ times.
- Each iteration executes 8 arithmetic operations.
- Hence, outer loop executes a total of $8(n^2-1)$ arithmetic operations.

Inner Loop

- Iterates $2K+4-2+1 = 2K+3$ times.
- Each iteration executes 8 arithmetic operations.
- 1^{st} inner loop (when $k=3$):
  - Iterates 9 times.
  - Hence, 1^{st} inner loop executes a total of $9(8) = 72$ arithmetic operations.
- Last inner loop (when $k=n^2+1$):
  - Iterates $2(n^2+1)+3 = 2n^2+2+3 = 2n^2+5$ times
  - Hence, last inner loop executes a total of $(2n^2+5)(8) = 16n^2+40$ arithmetic operations.
- Since the # of arithmetic operations executed by each inner loop correspond to an arithmetic sequence, the total # of arithmetic operations executed by all the inner loops $= \left(\frac{72+16n^2+40}{2}\right) * (n^2-1)$

$$\boxed{\text{Hence, the total number of arithmetic operations executed} = 8n^2-8+\left(\frac{16n^2+112}{2}\right)(n^2-1)}$$

(ii) Find the complexity of the code, i.e., O(code)

$$\boxed{O(code) = n^4}$$

QUESTION 2. (4 points) Is $(S_1 \Rightarrow S_2) \equiv (\overline{S_1} \vee S_2)$? Explain, i.e., construct the truth table.

S_1	S_2	$S_1 \Rightarrow S_2$	$\overline{S_1}$	$\overline{S_1} \vee S_2$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Since the two columns of the truth table $S_1 \Rightarrow S_2$ and $\overline{S_1} \vee S_2$ are the same, the expression $(S_1 \Rightarrow S_2$ is logically equivalent to $(\overline{S_1} \vee S_2$

QUESTION 3. (4 points) Given $a_n = a_{n-1} + 6a_{n-2}$, *such that* $a_1 = 10$, $a_2 = 10$

(a) Find a general formula for a_n.

$$a_n - a_{n-1} - 6a_{n-2} = 0 \Rightarrow \frac{\alpha^n - \alpha^{n-1} - 6\alpha^{n-2}}{\alpha^{n-2}} = 0$$

$$\Rightarrow \alpha^2 - \alpha + 6 = 0$$

$$\Rightarrow (\alpha-3)(\alpha+2) = 0 \Rightarrow \alpha_1 = 3, \alpha_2 = -2.$$

Hence, $a_n = c_1(3)^n + c_2(-2)^n$

① $a_1 = 3c_1 - 2c_2 = 10$

② $a_2 = 9c_1 + 4c_2 = 10$

Solving the two equations simultaneously using calculator, we obtain $c_1 = 2$ and $c_2 = -2$. $\boxed{a_n = 2(3)^n - 2(-2)^n}$

(b) Use (a) and find a_5.

$$a_5 = 2(3)^5 - 2(-2)^5 = \boxed{550}$$

QUESTION 4. (4 points) Given $a_n = 6a_{n-1} - 9a_{n-2}$, *such that* $a_1 = 9$, $a_2 = 36$. Find a general formula for a_n.

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \Rightarrow \frac{\alpha^n - 6\alpha^{n-1} + 9\alpha^{n-2}}{\alpha^{n-2}} = 0 \Rightarrow$$

$$\alpha^2 - 6\alpha + 9 = 0 \Rightarrow (\alpha-3)(\alpha-3) = 0$$

$\Rightarrow \alpha = 3$ repeated twice.

$\boxed{a_n = 2(3)^n + n(3)^n}$

Hence, $a_n = c_1(3)^n + c_2 n(3)^n$

① $a_1 = 3c_1 + 3c_2 = 9$

② $a_2 = 9c_1 + 18c_2 = 36$

Solving the two equations simultaneously using calculator, we obtain $c_1 = 2$, $c_2 = 1$

$A(n-1) = An - A$

$A(n-2) = An - 2A$

QUESTION 5. (6 points) Given $a_n = 4a_{n-1} - 3a_{n-2} + 1.25(2^n)$, *such that* $a_1 = 0$, $a_2 = 2$. Find a general formula for a_n.

$a_n = a_h + a_p$.

$a_h \Rightarrow a_n - 4a_{n-1} + 3a_{n-2} = 0 \Rightarrow \dfrac{\alpha^n - 4\alpha^{n-1} + 3\alpha^{n-2} = 0}{\alpha^{n-2}} \Rightarrow$

$\alpha^2 - 4\alpha + 3 = 0 \Rightarrow (\alpha-3)(\alpha-1) = 0$.

$\Rightarrow \alpha_1 = 3, \alpha_2 = 1$. Hence, $a_h = c_1(3)^n + c_2(1)^n$

$a_p \Rightarrow a_n - 4a_{n-1} + 3a_{n-2} = 1.25(2^n)$.

Since 2 is not a solution to a_h, $a_p(n) = A(2^n)$

$\Rightarrow a_p(n) - 4a_p(n-1) + 3a_p(n-2) = 1.25(2^n)$.

$\Rightarrow A(2^n) - 4A(2^{n-1}) + 3A(2^{n-2}) = 1.25(2^n)$

$\Rightarrow A(2^n) - \frac{4}{2}A(2^n) + \frac{3}{4}A(2^n) = 1.25(2^n)$.

$\Rightarrow A(2^n) - 2A(2^n) + \frac{3}{4}A(2^n) = 1.25(2^n)$.

$\Rightarrow A\cancel{(2^n)}\left[1 - 2 + \frac{3}{4}\right] = 1.25\cancel{(2^n)} \Rightarrow -\frac{1}{4}A = 1.25$ TBC

QUESTION 6. (6 points) Given $a_n = 3a_{n-1} - 2a_{n-2} + 5$, *such that* $a_1 = -1$, $a_2 = -4$. Find a general formula for a_n.

$a_n = a_h + a_p$

$a_h \Rightarrow a_n - 3a_{n-1} + 2a_{n-2} = 0 \Rightarrow \dfrac{\alpha^n - 3\alpha^{n-1} + 2\alpha^{n-2} = 0}{\alpha^{n-2}} \Rightarrow \alpha^2 - 3\alpha + 2 = 0$

$\Rightarrow (\alpha-2)(\alpha-1) = 0 \Rightarrow \alpha_1 = 2, \alpha_2 = 1$.

Hence, $a_h = c_1(2)^n + c_2(1)^n = c_1(2^n) + c_2$.

$a_p \Rightarrow a_n - 3a_{n-1} + 2a_{n-2} = 5$

Since 1 is a solution to the homogeneous, and 1 is repeated only once, $a_p(n) = An$

$\Rightarrow a_p(n) - 3a_p(n-1) + 2a_p(n-2) = 5 \Rightarrow An - 3(An - A) + 2(An - 2A) = 5$

$\Rightarrow \cancel{An} \cancel{-3An} + 3A \cancel{+2An} - 4A = 5 \Rightarrow 0An - A = 5 \Rightarrow -A = 5 \Rightarrow A = -5$

Hence, $a_p(n) = -5n$.

Thus, $a_n = c_1(2^n) + c_2 - 5n$.

① $a_1 = 2c_1 + c_2 - 5 = -1 \Rightarrow 2c_1 + c_2 = 4$

② $a_2 = 4c_1 + c_2 - 10 = -4 \Rightarrow 4c_1 + c_2 = 6$

Using calculator, $c_1 = 1$, $c_2 = 2$.

Hence, ~~$a_n = c_1(2)^n +$~~ $\boxed{\text{Hence, } a_n = (2)^n + 2 - 5n}$

$N^* = Z^+$

QUESTION 7. (10 points) Just write down T or F

(i) $\underbrace{\exists\ x \in Z \text{ such that } 3x+2=0}_{S_1 = F}$ if and only if $\underbrace{\exists\ y \in R \text{ such that } y^2+4=2}_{S_2 = F}$ T

$Z = \{\dots,-2,-1,0,1,2,\dots\}$

$3x+2=0 \Rightarrow x = -2/3 \notin Z$ $\quad y^2 = -2.\ y = \pm\sqrt{-2} = \pm\sqrt{2}i$

(ii) If $\underbrace{\exists!\ x \in N \text{ such that } x^2-2x-3=0}_{S_1 = T}$, then $\underbrace{\exists\ y \in R \text{ such that } 2y^2+1=5}_{S_2 = T}$ T

$\{0,1,2,\dots\}$

$(x-3)(x+1)=0 \Rightarrow x = 3$ $\quad 2y^2 = 4 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

(iii) $\forall x \in R,\ \exists!\ y \in R$ such that $x^2+y=-7$ T

$y = -7-x^2$

(iv) If $\underbrace{\exists\ x \in Z \text{ such that } x^2-0.25=0}_{S_1 = F}$, then $\underbrace{\forall\ y \in R, \text{ we have } y=-y+2022}_{S_2 =}$ T

$x^2 - \frac{1}{4} = (x-\frac{1}{2})(x+\frac{1}{2})$

(v) $\exists!\ x \in Z$ such that $\forall\ y \in R$, we have $2xy+4y=0$ T

$y(2x+4)=0$ $\begin{cases} 2x+4 \neq 0, \text{ then } y \text{ must be } 0. \\ 2x+4=0,\ y \text{ can be any } \#. \end{cases}$

$x=-2 \Rightarrow y(0)=0$

(vi) $\forall\ x \in Q^*,\ \exists!\ y \in Z^*$ such that $yx \in Z$. F

let $x = \frac{5}{4}$. $y = 4. \Rightarrow \frac{5}{4} \times 4 = 5 \in Z^*$ let $y=-4$. $\frac{5}{4} \times -4 = -5 \in Z^*$

(vii) $\exists\ x \in Q^*$ such that $\forall\ y \in Q^*$, we have $xy=1$ F

$x = \frac{1}{y}$

(viii) $\underbrace{\exists!\ x \in Z \text{ such that } 2x^2+3x=0}_{S_1 = T}$ if and only if $\underbrace{\exists!\ y \in N \text{ such that } y^2-3y+2=0}_{S_2 = F}$ F

$x(2x+3)=0.\ x=0, \cancel{x=-\frac{3}{2}}$ $\quad (y-2)(y-1)=0$

(ix) If $\underbrace{\exists!\ x \in N^* \text{ such that } x^3=x}_{S_1 = T}$, then $\underbrace{\exists!\ y \in Q^* \text{ such that } 2y^2=8}_{S_2 = F}$ F $\quad y=2,\ y=1$

$x^3 - x = 0 \Rightarrow x(x^2-1)=0$ $\quad y^2 = 4 \Rightarrow y = \pm 2$

$\cancel{x=0}$, $x=1$, $\cancel{x=-1}$

$x(x-1)(x+1)=0$

(x) $\underbrace{\exists\ x \in Q^* \text{ suchh that } x^2-4x=4}_{S_1 = T}$ if and only if $\underbrace{\exists\ y \in Z \text{ such that } 2y^2-6y=0}_{S_2 = T}$ T

-1

QUESTION 8. (4 points) As explained in the class, if S_1, then S_2 can be written as $S_1 \Rightarrow S_2$. Is $(S_1 \Rightarrow S_2) \vee (S_2 \Rightarrow S_1)$ a tautology statement? Explain, i.e., construct the truth table.

S_1	S_2	$S_1 \Rightarrow S_2$	$S_2 \Rightarrow S_1$	$(S_1 \Rightarrow S_2) \vee (S_2 \Rightarrow S_1)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

Yes, $(S_1 \Rightarrow S_2) \vee (S_2 \Rightarrow S_1)$ is a tautology statement since it always evaluates to true.